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# Similar Operators and a Functional Calculus for the First-Order Linear Differential Operator

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## 1. INTRODUCTION

The general problem of finding solutions of linear functional equations is undoubtedly one of the main problems of applied mathematics. Among such equations the classes of linear differential equations and linear difference equations are very important and numerous theories and methods for their solution have been developed. The methods based on integral transforms and the operational methods are certainly the ones used most often. In the present paper we present some general linear algebra results that may be considered as a unified approach to the transform and operational methods. We use the linear algebra foundations of the Laplace transform and the concept of similarity to generalize a description of the general solution of linear differential equations with constant coefficients in terms of certain basis for the space of exponential polynomials, and a convolution product defined in an algebraic way using the basis. The generalization gives an explicit construction of the general solution of any

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linear functional equation that is "similar" to a differential equation with constant coefficients.

Let $\mathscr{G}$ be a complex vector space and let $L: \mathscr{G} \to \mathscr{G}$ be a linear operator. Let $u(z)$ be a monic polynomial of positive degree and let $f$ be a given element of $\mathscr{G}$. Consider the problem of finding all the solutions $g$ in $\mathscr{G}$ of the linear equation,

$$u(L)g = f. \tag{1.1}$$

In Section 4 we show that solving (1.1) is quite easy provided that we can find a basis for $\mathscr{G}$ that is related in a certain way to the operator $L$. See Eq. (4.3). This is a way of saying that $L$ behaves as a generalized differentiation operator on the space $\mathscr{G}$.

The construction of the solutions of (1.1) requires the introduction of a convolution product in the space $\mathscr{G}$. Such a product is defined in a purely algebraic way in terms of the basic elements of $\mathscr{G}$. The convolution is used to construct a right inverse for the operator $u(L)$.

Our motivation comes from the linear algebra ideas that serve as a foundation for the transform methods for the solution of linear equations. In Section 2 we use a simplified algebraic version of the Laplace transform to illustrate how we use the idea of similarity in order to obtain the abstract setting for our main results. In Section 3 we present some elementary properties of polynomials and rational functions, related to partial fractions decomposition, and we use them to describe the general solution of an inhomogeneous linear differential equation with constant coefficients. The results of Section 3 are used as a model which we translate to an abstract setting in Section 4. We also give direct proofs of the main results in the abstract setting, without relying on similarity or particular properties of the vector space or the linear operators.

In Section 5 we use similarity to show that our general methods can be applied to a large class of linear differential equations with variable coefficients. In Section 6 we apply the methods to equations of the form $u(L_1)g = f$ where $L_1 = a(t)D + b(t)I$ is a linear differential operator of first order with variable coefficients. In Section 7 we present several concrete examples and some criteria to determine if a given second-order differential operator with variable coefficients can be written in the form $u(L_1)$. Among the examples we include the so-called binomial linear differential equation of order $n$, which is related to the Hermite polynomials.

The first part of this paper presents a significant simplification of the developments reported in our previous papers [6–9]. It also clarifies the role of convolutions in the classical transform and operational methods, and simplifies the construction of convolution products that relate in a nice way to a given operator, which is one of the main problems in Dimovski's book [3]. The second part contains generalizations of some results presented in [4].

## 2. SIMILARITY AND EQUIVALENT EQUATIONS

Let $X$ and $Y$ be nonempty sets and let $T\colon X \to Y$ be a bijective function. Let $f\colon X \to X$ be a function and let $b$ be a given element of $X$. Define $S = \{x \in X\colon\ f(x) = b\}$. Let $F\colon\ Y \to Y$ be defined by $F = T \circ f \circ T^{-1}$ and let $U = \{y \in Y\colon F(y) = T(b)\}$. Using only basic properties of the composition of functions it is easy to see that $S = T^{-1}(U)$ and $U = T(S)$. Therefore, finding the solution set $S$ is equivalent to finding the set $U$. In other words, solving $f(z) = b$ in the set $X$ is logically equivalent to solving $F(y) = T(b)$ in the set $Y$. In some concrete situations one of the two problems is considered easier than the other one. For example, in order to solve $f(x) = b$ one applies the transform $T$ to get the transformed equation $F(y) = T(b)$, which is considered easier to be solved. Then one finds somehow the solution set $U$, and finally, the set $S$ is obtained by applying the inverse transform $T^{-1}$ to the elements of $U$. This is the foundation of the transform methods, like the Laplace transform for differential equations and the $z$-transform for difference equations. Let us note that such ideas yield a useful general method only if the computation of images under $T$ and $T^{-1}$ can be done in a relatively simple way.

We use next an algebraic version of the Laplace transform to illustrate the main ideas of the present paper. Define the functions,

$$e_{a,k}(t) = \frac{t^k}{k!} e^{at}, \qquad a \in \mathbb{C},\ k \in \mathbb{N},$$

where $t$ is a complex variable, and let $\mathscr{E}$ be the complex vector space generated by the set of all the $e_{a,k}$, for $(a, k)$ in $\mathbb{C} \times \mathbb{N}$. The elements of $\mathscr{E}$ are called quasi-polynomials or exponential polynomials. Let $\mathscr{R}$ be the complex vector space spanned by the rational functions,

$$r_{a,k}(z) = \frac{1}{(z-a)^{1+k}}, \qquad a \in \mathbb{C},\ k \in \mathbb{N}.$$

The elements of $\mathscr{R}$ are called proper rational functions. By the division algorithm for polynomials the space of all the rational functions is the direct sum $\mathscr{P} \oplus \mathscr{R}$, where $\mathscr{P}$ denotes the space of all polynomials in one variable. The bijective map $T$ between the bases of $\mathscr{E}$ and $\mathscr{R}$, defined by $Te_{a,k} = r_{a,k}$, extends by linearity to a vector space isomorphism $T\colon \mathscr{E} \to \mathscr{R}$.

The usual differentiation operator $D$ satisfies

$$De_{a,k} = \begin{cases} ae_{a,0}, & \text{if } k = 0, \\ ae_{a,k} + e_{a,k-1}, & \text{if } k \geq 1. \end{cases} \tag{2.1}$$

Let us define the linear operator $H$ on the space $\mathscr{R}$ by

$$Hr_{a,k} = \begin{cases} ar_{a,0}, & \text{if } k = 0, \\ ar_{a,k} + r_{a,k-1}, & \text{if } k \geq 1. \end{cases} \tag{2.2}$$

It is clear that $D = T^{-1}HT$. Thus for any polynomial $u$ the differential operator $u(D)$, which acts on the space $\mathscr{E}$, is similar to the operator $u(H)$, which acts on $\mathscr{R}$. A trivial computation yields $Hr_{a,k}(z) = zr_{a,k}(z)$, for $k \geq 1$, and $Hr_{a,0}(z) = zr_{a,0} - 1$. This shows that the action of $H$ on the proper rational functions is multiplication by $z$ followed by projection on the space $\mathscr{R}$. Thus, for $g$ in $\mathscr{R}$ and $k \geq 0$, $H^k g(z)$ is the proper rational part of the rational function $z^k g(z)$. In other words, $H^k g(z)$ is $z^k g(z)$ reduced modulo the polynomials.

Let $u$ be a monic polynomial of positive degree and let $f$ be a given element of $\mathscr{E}$. Then, solving the differential equation,

$$u(D)y = f, \tag{2.3}$$

in the space $\mathscr{E}$ is equivalent to finding the proper rational functions $g$ that satisfy the equation $u(H)g(z) = Tf$, which may be written in the form,

$$u(z)g(z) \equiv Tf \pmod{\mathscr{P}}. \tag{2.4}$$

It is clear that the set of solutions of (2.4) is

$$U = \left\{ g = \frac{Tf}{u} + \frac{p}{u} : p \in \mathscr{P} \text{ and } \frac{p}{u} \in \mathscr{R} \right\}. \tag{2.5}$$

Notice that because $Tf$ is in $\mathscr{R}$ then $Tf/u$ is also in $\mathscr{R}$. Note also that $p/u$ is in $\mathscr{R}$ if and only if $p$ is a polynomial whose degree is strictly smaller than the degree of $u$.

The usual Laplace transform method gives the set of solutions of (2.3) in the form $S = T^{-1}(U)$. The main tool for the computation of images under $T^{-1}$ is the partial fractions decomposition formula. Once we have an element $g$ of $U$ written as a linear combination of basic proper rational functions, applying $T^{-1}$ is just substitution of each $r_{a,k}$ that appears in the expression for $g$ by the corresponding basic exponential polynomial $e_{a,k}$.

Let us note that the procedure described previously to find the solutions of (2.3) does not require the representation of the map $T$ as an integral transform. Note also that one of the main limitations to the generality of Eq. (2.3) is the condition that the forcing function $f$ must be an element of $\mathscr{E}$. This limitation will be eliminated in the next section. We will find first a description of the construction of the set $U$ in terms of operations with basic proper rational functions, and then use similarity to translate the construction to one that can be done entirely in the space $\mathscr{E}$, without using the maps $T$ and $T^{-1}$. The logical equivalence of Eqs. (2.3) and (2.4) makes this objective possible.

## 3. PARTIAL FRACTIONS AND THE CONVOLUTION ON $\mathscr{E}$

In this section we introduce some definitions and basic properties of rational functions that will be used in the rest of the paper.

In order to express a proper rational function as a linear combination of basic functions $r_{a,k}$, we need the partial fractions decomposition formula, which we discuss next.

Let

$$u(z) = \prod_{j=0}^{d} (z - a_j)^{m_j}, \tag{3.1}$$

where the $a_j$ are distinct complex numbers, the $m_j$ are positive integers, and the degree of $u$ is $\sum m_j = n + 1$.

Define the polynomials

$$q_{j,m_j-1-k}(z) = \frac{u(z)}{(z-a_j)^{1+k}}, \qquad 0 \le j \le d, 0 \le k \le m_j - 1. \tag{3.2}$$

Let $T_{a,k}$ denote the Taylor functional defined by

$$T_{a,k} f(z) = \frac{D^k f(a)}{k!}, \qquad (a,k) \in \mathbb{C} \times \mathbb{N}, \tag{3.3}$$

where $f$ is any function for which the right-hand side is well defined. The linear functionals $Q_{j,k}$ on the space $\mathscr{P}$ are defined by

$$Q_{j,k} p(z) = T_{a_j,k} \frac{p(z)}{q_{j,0}(z)}, \qquad 0 \le j \le d, 0 \le k \le m_j - 1. \tag{3.4}$$

A simple computation using Leibniz's rule yields the biorthogonality relation,

$$Q_{j,k} q_{i,m} = \delta_{(j,k),(i,m)}. \tag{3.5}$$

Therefore the polynomials $q_{i,m}$ form a basis for the vector space $\mathscr{P}_n$ of the polynomials with degree at most equal to $n$, and the functionals $Q_{j,k}$ form a basis for the dual space of $\mathscr{P}_n$. Consequently, for every polynomial $p$ in $\mathscr{P}_n$ we have

$$p(z) = \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} Q_{j,k} p q_{j,k}(z). \tag{3.6}$$

Dividing by $u(z)$ and reordering the terms in the inner sum we get the *partial fractions decomposition formula*,

$$\frac{p(z)}{u(z)} = \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \frac{Q_{j,m_j-1-k} p}{(z-a_j)^{1+k}}. \tag{3.7}$$

Because the elements of the set $U$, defined in (2.5), are of the form $g = (Tf)/u + p/u$, where $p$ is in $\mathscr{P}_n$, the partial fractions decomposition formula (PFD) gives us $p/u$ as a linear combination of basic proper rational functions. The term $(Tf)/u$ is a product of elements of $\mathscr{R}$. The PFD can be used to express it as a linear combination of basic rational functions.

Taking $u(z) = (z-a)^{1+k}(z-b)^{1+m}$, with $a \neq b$, and applying the PFD we get the multiplication formula,

$$r_{a,k} r_{b,m} = \sum_{j=0}^{k} C(a,j;b,m) r_{a,k-j} + \sum_{j=0}^{m} C(b,j;a,k) r_{b,m-j}, \tag{3.8}$$

where the coefficients are defined by

$$C(a,j;b,i) = (-1)^j \binom{j+i}{j} (a-b)^{-1-j-i}, \qquad a \neq b,\ i,j \in \mathbb{N}. \tag{3.9}$$

Notice that $r_{a,k} r_{a,m} = r_{a,1+k+m}$.

Taking $p = 1$ in the PFD formula (3.7) we get

$$\frac{1}{u(z)} = \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \alpha_{j,k} r_{a_j,k}(z), \tag{3.10}$$

where

$$\alpha_{j,k} = Q_{j,m_j-1-k} 1. \tag{3.11}$$

Note that the coefficients $\alpha_{j,k}$ depend only on the roots of $u(z)$ and their multiplicities.

Multiplying (3.10) by $u(z)$ and using (3.2) we get the polynomial identity,

$$\sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \alpha_{j,k} q_{j,m_j-1-k}(z) = 1. \tag{3.12}$$

Now we can describe the construction of the elements of the set $U$ in terms of basic proper rational functions. Because $f$ is in $\mathscr{E}$ it is a finite linear combination of functions of the form $e_{b,j}(t)$ and hence $Tf$ is a linear combination of functions $r_{b,j}(z)$. Therefore every solution $g$ of Eq. (2.4) has the form,

$$g(z) = \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \alpha_{j,k} r_{a_j,k}(z) Tf(z) + \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \beta_{j,k} r_{a_j,k}(z), \tag{3.13}$$

where the coefficients $\beta_{j,k}$ are arbitrary complex numbers.

We want to transfer (3.13) to the corresponding equation for $T^{-1}g$ in the space $\mathscr{E}$. Instead of performing first the multiplication in the first term of (3.13) and then apply $T^{-1}$, we can define an operation on $\mathscr{E}$ that corresponds to multiplication in $\mathscr{R}$. The convolution $*$ on $\mathscr{E}$ is defined by

$$e_{a,k} * e_{b,m} = \sum_{j=0}^{k} C(a,j;b,m) e_{a,k-j} + \sum_{j=0}^{m} C(b,j;a,k) e_{b,m-j}, \tag{3.14}$$

where the coefficient functions are defined in (3.9). This definition clearly implies

$$f * h = T^{-1}(Tf Th), \qquad f, h \in \mathscr{E}, \tag{3.15}$$

where the multiplication in the right-hand side is multiplication of rational functions. Now we can apply $T^{-1}$ to (3.13) to obtain the solutions of Eq. (2.3). In this way we get the following.

THEOREM 3.1. *The general solution of the differential equation $u(D)y(t) = f(t)$, where $f$ is a given element of $\mathscr{E}$ is*

$$y(t) = f(t) * h_u(t) + \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \beta_{j,k} e_{a_j,k}(t), \tag{3.16}$$

*where*

$$h_u(t) = \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \alpha_{j,k} e_{a_j,k}(t),$$

*the $\alpha_{j,k}$ are defined in* (3.11), *and the $\beta_{j,k}$ are arbitrary complex numbers.*

In order to compute the right-hand side of (3.16) we only need the representation of $f$ as a linear combination of basic exponential polynomials, the determination of the numbers $\alpha_{j,k}$, which depend only on the polynomial $u$, and the convolution formula (3.14). All of this can now be done without ever mentioning the map $T$.

Notice that the last term in (3.16) is the general solution of the homogeneous equation $u(D)y = 0$, and the other term is a particular solution of the inhomogeneous equation $u(D)y = f$. Note also that $h_u = T^{-1}(1/u)$.

COROLLARY 3.1. *The linear map on the space $\mathscr{E}$ that sends $f$ to $f * h_u$ is a right inverse for the operator $u(D)$.*

## 4. THE ABSTRACT SETTING

In this section we use the results about the space $\mathscr{E}$ of exponential polynomials presented in the previous sections as a model and we obtain extensions of the main results in a very general setting. Although the use of similarity gives an almost trivial proof of our results, we prefer to give direct proofs that use only basic properties of polynomials and rational functions, and which are also useful to obtain important generalizations.

Let $\mathscr{G}$ be a complex vector space that has a basis $\{g_{a,k} : (a,k) \in \mathbb{C} \times \mathbb{N}\}$. We define the commutative convolution product $*$ on $\mathscr{G}$ as follows. If $a \neq b$,

$$g_{a,k} * g_{b,m} = \sum_{j=0}^{k} C(a,j;b,m) g_{a,k-j} + \sum_{j=0}^{m} C(b,j;a,k) g_{b,m-j}, \quad (4.1)$$

where the coefficient functions are defined in (3.9), and

$$g_{a,k} * g_{b,m} = g_{a,1+k+m}, \qquad k,m \in \mathbb{N}. \quad (4.2)$$

The linear map $L: \mathscr{G} \to \mathscr{G}$ is defined by

$$Lg_{a,k} = \begin{cases} ag_{a,0}, & \text{if } k = 0, \\ ag_{a,k} + g_{a,k-1}, & \text{if } k \geq 1. \end{cases} \quad (4.3)$$

For any $a$ in $\mathbb{C}$ it is obvious that

$$(L - aI) g_{a,k} = \begin{cases} 0, & \text{if } k = 0, \\ g_{a,k-1}, & \text{if } k \geq 1, \end{cases} \quad (4.4)$$

and thus,

$$(L - aI)^{1+m} g_{a,k} = 0, \qquad m \geq k. \quad (4.5)$$

By induction on $m$ it is easy to see that

$$(L - aI)^m g_{b,k} = \sum_{j=0}^{s} \binom{m}{j} (b-a)^{m-j} g_{b,k-j}, \tag{4.6}$$

where $s = \min\{m, k\}$. Therefore, for $a \neq b$ we have

$$(L - aI)^m g_{b,k} \neq 0, \qquad m, k \in \mathbb{N}. \tag{4.7}$$

A straightforward computation yields

$$L(g_{a,k} * g_{b,m}) = (Lg_{a,k}) * g_{b,m} + g_{b,m} \Phi g_{a,k}, \tag{4.8}$$

where $\Phi$ is the linear functional on $\mathscr{G}$ defined by $\Phi g_{a,k} = \delta_{0,k}$. By linearity we obtain

$$L(g * f) = (Lg) * f + f \Phi g, \qquad f, g \in \mathscr{G}. \tag{4.9}$$

Using induction it is easy to prove that

$$(L - aI)^{1+k}(g_{a,k} * f) = f, \qquad (a,k) \in \mathbb{C} \times \mathbb{N}, f \in \mathscr{G}. \tag{4.10}$$

Let $u(z)$ be a monic polynomial of positive degree as in (3.1). Then, by (3.10), the PFD for $1/u$ is

$$\frac{1}{u(z)} = \sum_{j=0}^{d} \sum_{k=0}^{m_j - 1} \alpha_{j,k} r_{a_j,k}(z).$$

We define

$$h_u = \sum_{j=0}^{d} \sum_{k=0}^{m_j - 1} \alpha_{j,k} g_{a_j,k}. \tag{4.11}$$

Translating by similarity Theorem 3.1 to the present setting we obtain

THEOREM 4.1. *Let $f$ be a given element of $\mathscr{G}$ and let the polynomial $u$ be as in the previous text. Then the general solution of the equation $u(L)g = f$ is*

$$g = h_u * f + \sum_{j=0}^{d} \sum_{k=0}^{m_j - 1} \beta_{j,k} g_{a_j,k}, \tag{4.12}$$

*where $h_u$ is defined in* (4.11) *and the coefficients $\beta_{j,k}$ are arbitrary complex numbers.*

*Proof.* The definition (3.2) of the polynomials $q_{j,k}$ gives

$$u(z) = q_{j,m_j-1-k}(z)(z-a_j)^{1+k}, \qquad 0 \le j \le d, 0 \le k \le m_j - 1,$$

and thus we have the factorizations

$$u(L) = q_{j,m_j-1-k}(L)\big(L - a_j I\big)^{1+k}, \qquad 0 \le j \le d, 0 \le k \le m_j - 1. \tag{4.13}$$

Then, by (4.5) we have

$$u(L) g_{a_j,k} = 0, \qquad 0 \le j \le d, 0 \le k \le m_j - 1, \tag{4.14}$$

and by (4.10),

$$\big(L - a_j I\big)^{1+k}\big(g_{a_j,k} * f\big) = f.$$

Let us note that (4.7) and (4.14) imply that $\{g_{a_j,k}\colon 0 \le j \le d, 0 \le k \le m_j - 1\}$ is a basis for the kernel of $u(L)$. Therefore for any $g$ given by (4.12) we have $u(L)g = u(L)(h_u * f)$ and hence,

$$\begin{aligned} u(L)(h_u * f) &= \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \alpha_{j,k} u(L)\big(g_{a_j,k} * f\big) \\ &= \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \alpha_{j,k} q_{j,m_j-1-k}(L)\big(L - a_j I\big)^{1+k}\big(g_{a_j,k} * f\big) \\ &= \sum_{j=0}^{d} \sum_{k=0}^{m_j-1} \alpha_{j,k} q_{j,m_j-1-k}(L) f \\ &= If. \end{aligned}$$

We have used here the polynomial identity (3.12). ∎

Suppose now that the convolution $*$ can be extended to a commutative bilinear map $*$ from $\mathscr{G} \times \mathscr{K}$ to $\mathscr{K}$, where $\mathscr{K}$ is a complex vector space that contains $\mathscr{G}$. Let us note that (4.9) allows us to extend the definition of the operator $L$ to the set of elements of the form $g * f$, where $g$ is in $\mathscr{G}$ and $f$ is in $\mathscr{K}$. From the proof of the theorem it is clear that the computation that yields $u(L)(h_u * f) = f$ does not require the application of $L$ to $f$. If $g$ is an element of $\mathscr{G}$ such that $\Phi g = 0$ and $f$ is in $\mathscr{K}$ then $L^m(g * f) = (L^m g) * f$ and thus,

$$w(L)(g * f) = \big(w(L) g\big) * f, \qquad w \in \mathscr{P}.$$

By a property of the residues of rational functions, for any polynomial $u$ with degree greater than or equal to 2 we have $\Phi(1/u) = 0$ and hence $\Phi h_u = 0$. Because $L(g_{a,0} * f) = ag_{a,0} * f + f$ it is clear that $L^2(g_{a,0} * f)$ is defined if and only if $Lf$ is defined. These observations about the extensions of $*$ show that we can we can generalize Theorem 4.1 and we can obtain the following.

COROLLARY 4.1. *Let $f$ be a given element of $\mathscr{K}$ and let $u$ be as in the preceding text. Then the general solution of the equation $u(L)g = f$ is*

$$g = h_u * f + \sum_{j=0}^{d} \sum_{k=0}^{m_j - 1} \beta_{j,k} g_{a_j,k}, \tag{4.15}$$

*where $h_u$ is as earlier and the $\beta_{j,k}$ are arbitrary numbers.*

Note that the element $h_u$ is a sort of "Green's function" for the operator $u(L)$.

The familiar example of the space of exponential polynomials will be used to illustrate how the convolution can be extended to a larger space. Let $\mathscr{G}$ denote the space of exponential polynomials of a real variable $t$. For $f$ and $g$ in $\mathscr{G}$ the *Duhamel convolution* $*$ is defined by

$$f * g(t) = \int_0^t f(x) g(t - x)\, dx. \tag{4.16}$$

In [6, Theorem 4.2] we proved that the convolution on the space of exponential polynomials defined in (3.14) coincides with the Duhamel convolution. Let us denote by $\mathscr{K}$ the vector space of all the piecewise continuous functions of the real variable $t$. It is clear that $\mathscr{G}$ is contained in $\mathscr{K}$ and that the convolution defined in (4.16) is well defined for $f$ and $g$ in $\mathscr{K}$. Therefore Corollary 4.1 holds and (4.15) gives the general solution of equations of the form $u(D)g = f$, where $f$ is an element of $\mathscr{K}$. Notice that in this case $\Phi f = f(0)$, for $f$ in $\mathscr{K}$. Note that the integral representation (4.16) is what allows us to extend the convolution to a larger space of functions.

The approach that we use in this section can be summarized as follows. We begin with a vector space $\mathscr{G}$ that has a basis $\{g_{a,k}\}$ and then we define in (4.3) the operator $L$ in terms of its action on the basis. Another approach is to start with a linear operator $L$ that acts on certain space of functions of a variable $t$ and then to construct the space $\mathscr{G}$ and the basis $\{g_{a,k}\}$ for which (4.3) holds. One way to do this is to find a function $G(z,t)$ of two variables, defined on some suitable domain, that satisfies

$$LG(z,t) = zG(z,t), \tag{4.17}$$

where $L$ acts with respect to $t$, considering $z$ as a parameter. Then it is easy to see that the functions,

$$g_{a,k}(t) = T_{a,k}G(z,t), \qquad (a,k) \in \mathbb{C} \times \mathbb{N}, \tag{4.18}$$

where the Taylor functional acts with respect to $z$, satisfy condition (4.3) and generate a vector space $\mathscr{G}$. The function $G(z,t)$ is called a generating function for the operator $L$. For example, for the differential operator $D$ a generating function is $\exp(zt)$, which yields the basic functions $e_{a,k}$. Using generating functions we can study other kinds of operators, like difference operators, for example. See our previous papers [6, 7, and 9].

## 5. SIMILAR OPERATORS AND CONVOLUTIONS

Using the space of exponential polynomials and the concept of similarity we can construct many other interesting examples. Suppose $A: \mathscr{H} \to \mathscr{G}$ is a linear isomorphism from a space $\mathscr{H}$, of functions defined on some subset of the real line, onto the space $\mathscr{G}$ of real exponential polynomials. Then the operator $L = A^{-1}DA$ maps $\mathscr{H}$ into $\mathscr{H}$ and the functions $h_{a,k} = A^{-1}e_{a,k}$ form a basis for $\mathscr{H}$. Denote by $\odot$ the convolution on $\mathscr{H}$. Then we have

$$h \odot f = A^{-1}(Ah * Af), \qquad h, f \in \mathscr{H}, \tag{5.1}$$

where $*$ denotes the convolution on $\mathscr{G}$. Note that for each appropriate operator $A$ Eq. (5.1) gives also an integral representation for $\odot$. It is clear from the definitions that

$$L(h \odot f) = (Lh) \odot f + f\Phi h, \qquad h, f \in \mathscr{H}, \tag{5.2}$$

where the functional $\Phi$ on $\mathscr{H}$ is defined by $\Phi h = \Phi(Ah) = (Ah)(0)$, because the functional $\Phi$ on $\mathscr{G}$ is evaluation at zero. We will use the same symbol $\Phi$ to denote the corresponding functional on any space isomorphic to $\mathscr{E}$.

We consider next an important class of examples related to the idea of change of variables in differential equations. Let $\beta(t)$ be a function defined on an interval $J = (c,d)$ that contains zero, such that $\beta(t)$ has an inverse under composition that we denote by $\tilde{\beta}(t)$. Let $\alpha(t)$ be a function defined on $J$ such that $\alpha(t) \neq 0$ for $t$ in $J$. Denote by $S_\beta$ the operator of substitution of $t$ by $\beta(t)$, and by $M_\alpha$ the operator of multiplication by $\alpha(t)$. Then the operator $L$ defined by

$$L = M_\alpha^{-1} S_{\tilde{\beta}} D S_\beta M_\alpha \tag{5.3}$$

is similar to the differential operator $D$. In this case the basis for the space $\mathscr{H}$ is the set of functions,

$$h_{a,k} = M_\alpha^{-1} S_{\tilde{\beta}} e_{a,k} = \frac{1}{\alpha(t)} \frac{\left(\tilde{\beta}(t)\right)^k}{k!} \exp\left(a\tilde{\beta}(t)\right). \tag{5.4}$$

From (5.1) we see that the convolution $\odot$ associated with the operator $L$ is

$$f \odot h = M_\alpha^{-1} S_{\tilde{\beta}} \left\{ (S_\beta M_\alpha f) * (S_\beta M_\alpha h) \right\},$$

where $*$ is the Duhamel convolution. Therefore,

$$(f \odot h)(t) = \frac{1}{\alpha(t)} \int_{\beta(0)}^{t} \alpha\left(\beta\left(\tilde{\beta}(t) - \tilde{\beta}(y)\right)\right) f\left(\beta\left(\tilde{\beta}(t) - \tilde{\beta}(y)\right)\right)$$
$$\times\, \alpha(y) h(y) \tilde{\beta}'(y)\, dy. \tag{5.5}$$

This integral representation can be used to define the convolution $\odot$ of functions in a space much larger than $\mathscr{H}$.

Using some basic properties of the Duhamel convolution it is easy to prove the following.

PROPOSITION 5.1. *Let $\alpha$ and $\beta$ be as in the foregoing text, let $f$ be a piecewise continuous function on the interval $J = (c, d)$, and let $h$ be a continuously differentiable function on $J$. Then we have*

$$L(f \odot h)(t) = f(t) \odot Lh(t) + f(t) h(\beta_0) \alpha(\beta_0), \tag{5.6}$$

*where $\beta_0 = \beta(0)$.*

Notice that we can apply Corollary 4.1 to find the solutions of equations of the form $u(L)g = f$, where $L$ is an operator of the form (5.3) and $f$ is as in Proposition 5.1.

## 6. GENERAL LINEAR DIFFERENTIAL OPERATOR OF THE FIRST ORDER

In this section we consider some elementary properties of the general linear first-order differential operator $L_1$, defined by

$$L_1 = a(t)D + b(t)I, \tag{6.1}$$

where $a(t)$ and $b(t)$ are continuous functions on an interval $J = (c, d)$, and $a(t) \neq 0$ for $t$ in $J$. A simple calculation yields the following.

PROPOSITION 6.1. *Let $r(t)$ and $s(t)$ be differentiable functions defined on $J$ such that*

$$r'(t) = \frac{b(t)}{a(t)}, \qquad s'(t) = \frac{1}{a(t)}. \tag{6.2}$$

*Then the function $G(z,t) = \exp(zs(t) - r(t))$ satisfies*

$$L_1 G(z,t) = zG(z,t). \tag{6.3}$$

From the generating function $G(z,t)$ we obtain the functions,

$$g_{a,k}(t) = T_{a,k}G(z,t) = e^{-r(t)}\frac{(s(t))^k}{k!}e^{as(t)}, \qquad (a,k) \in \mathbb{C} \times \mathbb{N}. \tag{6.4}$$

These functions generate a vector space $\mathscr{G}$ and satisfy

$$L_1 g_{a,k} = \begin{cases} ag_{a,0}, & \text{if } k = 0, \\ ag_{a,k} + g_{a,k-1}, & \text{if } k \geq 1. \end{cases} \tag{6.5}$$

Therefore we can apply Theorem 4.1 to solve equations of the form $u(L_1)g = f$, for $f$ in $\mathscr{G}$. We show next that $L_1$ is similar to the differential operator $D$.

If $L$ is an operator similar to $D$ given by (5.3) then a simple computation gives

$$Ly(t) = \beta'(\tilde{\beta}(t))y'(t) + \beta'(\tilde{\beta}(t))\frac{\alpha'(t)}{\alpha(t)}y(t),$$

that is

$$L = \frac{1}{\tilde{\beta}'(t)}D + \frac{\alpha'(t)}{\alpha(t)\tilde{\beta}'(t)}I.$$

From this expression and (6.1), we obtain immediately the following.

PROPOSITION 6.2. *Let $L_1$ be the differential operator defined by* (6.1) *and let $r(t)$ and $s(t)$ be functions that satisfy* (6.2). *Suppose that $s(t)$ has an inverse under composition denoted by $\tilde{s}(t)$. Then $L_1$ can be written as*

$$L_1 = M_\alpha^{-1} S_{\tilde{\beta}} D S_\beta M_\alpha,$$

*where $\beta(t) = \tilde{s}(t)$ and $\alpha(t) = e^{r(t)}$.*

Therefore the operator $L_1$ is similar to $D$.

COROLLARY 6.1. *If* $s(t_0) = 0$ *for some* $t_0$ *in* $J$, *then the convolution* $\odot$ *for the operator* $L_1$ *is*

$$(f \odot h)(t) = e^{-r(t)} \int_{t_0}^{t} \exp\{r(\tilde{s}(s(t) - s(y)))\} \times f(\tilde{s}(s(t) - s(y))) e^{r(y)} h(y) s'(y)\, dy, \tag{6.6}$$

*and*

$$L_1(f \odot h)(t) = f(t) \odot L_1 h(t) + f(t) h(t_0) e^{r(t_0)}, \tag{6.7}$$

*for* $h$ *in* $\mathscr{G}$ *and some appropriate function* $f$.

## 7. SOME EXAMPLES

Let $u(z) = z^2 + b_1 z + b_2$ be a monic polynomial of degree 2 and let $L_1$ be the differential operator defined by (3.1). Then

$$u(L_1) = a^2(t) D^2 + a(t)\{a'(t) + 2b(t) + b_1\} D + \{b^2(t) + a(t) b'(t) + b_1 b(t) + b_2\} I. \tag{7.1}$$

Using this equation we characterize two types of differential equations that can be solved by the method described in the previous sections.

First, if we put $a(t) = 1$ in (7.1) and we compare it with the linear differential equation of second order,

$$y''(t) + p(t) y'(t) + q(t) y(t) = 0, \tag{7.2}$$

we obtain the following.

PROPOSITION 7.1. *The differential Eq.* (7.2) *can be written in the form* $u(L) = 0$, *where* $u(z) = z^2 + b_1 z + b_2$, $L = D + b(t)$, *and* $b(t) = (p(t) - b_1)/2$, *if*

$$p^2(t) + 2p'(t) - 4q(t) = b_1^2 - 4b_2.$$

In an analogous way, taking $b(t) = 0$ in (7.1) we get

PROPOSITION 7.2. *Let* $L = a(t)D$ *and let* $u(z) = z^2 + b_1 z + b_2^2$ *be a monic polynomial of degree* 2 *with* $b_2 \neq 0$. *A necessary and sufficient condition for the equation,*

$$y''(t) + f(t) y'(t) + g^2(t) y(t) = 0,$$

*to be of the form* $u(L) = 0$ *is*

$$\frac{f(t)g(t) + g'(t)}{g^2(t)} = \frac{b_1}{b_2}.$$

*In such case* $a(t) = b_2/g(t)$.

EXAMPLE 1. For $\alpha, \beta$ in $\mathbb{C}$, let us consider the differential equation,

$$y'' + (2\alpha t + \beta)y' + \alpha t(\alpha t + \beta)y = 0, \qquad +\infty < t < \infty. \tag{7.3}$$

In this case $p(t) = 2\alpha t + \beta$, $q(t) = \alpha t(\alpha t + \beta)$, and

$$p^2(t) + 2p'(t) - 4q(t) = 4\alpha + \beta^2.$$

Therefore it is possible to apply Proposition 7.1 to solve the equation. In order that $b_1^2 - 4b_2 = 4\alpha + \beta^2$, we can select the values of $b_1$ and $b_2$ in several ways. Let $b_1 = 0$, $b_2 = -(\alpha + \beta^2/4)$, $u(z) = z^2 - (\alpha + \beta^2/4)$, and $L = D + (\alpha t + \beta/2)$. Then Eq. (7.3) takes the form $u(L)y = 0$. Therefore its general solution is given by

$$y(t) = C_1 \exp\left\{a_0 t - \frac{1}{\alpha}\left(\alpha t + \frac{\beta}{2}\right)^2\right\} + C_2 \exp\left\{a_1 t - \frac{1}{\alpha}\left(\alpha t + \frac{\beta}{2}\right)^2\right\},$$

where $a_0, a_1$ are the roots of the polynomial $u(z)$, and $C_1$ and $C_2$ are arbitrary complex numbers.

EXAMPLE 2. For the differential equation,

$$ty'' + (t^2 - 1)y' + t^3 y = 0, \qquad 0 < t < \infty, \tag{7.4}$$

if we let $f(t) = (t^2 - 1)/t$, and $g(t) = t$, then

$$\frac{f(t)g(t) + g'(t)}{g^2(t)} = 1.$$

This means that the hypothesis of Proposition 7.2 is satisfied. We choose $b_1$ and $b_2$ such that $b_1/b_2 = 1$, for example, $b_1 = b_2 = 2$. Then the differential equation can be written as $u(L)y = 0$, with $L = (1/t)D$ and $u(z) = z^2 + 2z + 4$. Therefore the general solution of (7.4) is

$$y(t) = C_1 \exp\left(a_0 \frac{t^2}{2}\right) + C_2 \exp\left(a_1 \frac{t^2}{2}\right),$$

where $a_0 = -1 - \sqrt{3}i$, $a_1 = -1 + \sqrt{3}i$, and $C_1, C_2$ are any complex numbers. Consider now the inhomogeneous equation,

$$y'' + \frac{(t^2 - 1)}{t} y' + t^2 y = t^2, \qquad t > 0. \tag{7.5}$$

For the operator $L = (1/t)D$ the corresponding convolution is

$$(h * F)(t) = \int_0^t F\left(\sqrt{t^2 - y^2}\right) h(y)\, y\, dy. \tag{7.6}$$

Because $u(z) = z^2 + 2z + 4$, a simple computation gives

$$h_u(t) = \mu \left\{ \exp\left( \frac{a_1}{2} t^2 \right) - \exp\left( \frac{a_0}{2} t^2 \right) \right\},$$

where $\mu = -(\sqrt{3}/6)i$. Let $F(t) = t^2$, the right-hand side of (7.5). Then a particular solution for (7.5) is

$$(h_u * F)(t) = \mu \int_0^t (t^2 - y^2) \left\{ \exp\left( \frac{a_1}{2} t^2 \right) - \exp\left( \frac{a_0}{2} t^2 \right) \right\} y\, dy.$$

Using integration by parts it is not difficult to obtain

$$(h_u * F)(t) = \frac{1}{4}(t^2 - 1) + \frac{\mu}{8} \left\{ a_0^2 \exp\left( \frac{a_1}{2} t^2 \right) - a_1^2 \exp\left( \frac{a_0}{2} t^2 \right) \right\}.$$

In a similar way we can solve the following differential equations,

$$\begin{aligned} y'' + ty' + e^{-t^2} y &= 0, \\ y'' + (\tan t) y' + \left( \frac{3}{4} \sec^2 t \right) y &= 0, \\ y'' + \frac{1}{t} y' - \frac{1}{4t^2} y &= 0, \\ y'' + (\cos t) y' - \frac{1}{4} \sin t (\sin t + 2) y &= 0. \end{aligned}$$

For more examples of this kind see [4].

EXAMPLE 3. We consider now a class of linear differential equations that are easily recognized. For $(c, n)$ in $\mathbb{C} \times \mathbb{N}$, the equation,

$$\sum_{k=0}^{n} \binom{n}{k} (ct)^k D^{n-k} y = 0 \tag{7.7}$$

is called the *binomial linear differential equation of order* $n$. This equation has been studied by several authors. See [1, 5, and 10].

Let $He_n(z; c)$ denote the Chebyshev–Hermite polynomials, defined by

$$He_n(z; c) = \sum_{k=0}^{[n/2]} \frac{n!(-c)^k}{2^k (n-2k)!k!} z^{n-2k}.$$

Then we can prove the identity,

$$He_n(D + ctI; c) = \sum_{k=0}^{n} \binom{n}{k} (ct)^k D^{n-k}, \tag{7.8}$$

which generalizes some operator identities of Klamkin [5] and Chatterjea [1]. This means that the differential operator of (7.7) is a polynomial in the operator $L_1 = D + ctI$. By Proposition 6.1 the generating function for $L_1$ is

$$G(z, t, c) = \exp\left( zt - \frac{c}{2} t^2 \right),$$

and hence the functions $g_k(t) = G(\mu_k, t, c)$, for $k = 1, 2, \ldots, n$, constitute a fundamental system of solutions for the differential equation (7.7), where $\mu_1, \mu_2, \ldots, \mu_n$, are the zeros of the Chebyshev–Hermite polynomial $He_n(z; c)$, which are distinct.

Let us note that any identity of the form,

$$u(a(t)D + b(t)I) = \sum_{k=0}^{n} f_k(t) D^{n-k}, \tag{7.9}$$

where $u$ is a polynomial and $a$, $b$, and $f_k$ are functions of $t$, allows us to solve differential equations of the form $w(L)y = 0$, where $L$ is the operator in the right-hand side of Eq. (7.9) and $w$ is a polynomial. Several identities of this kind have been obtained in [2] and [11].

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